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Tridiagonal matrices, orthogonal polynomials and Diophantine relations: I

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Abstract

It is well known that the eigenvalues of tridiagonal matrices can be identified with the zeros of polynomials satisfying three-term recursion relations and being therefore members of an orthogonal set. A class of such polynomials is identified some of which feature zeros given by simple formulae involving integer numbers. In the process certain neat formulae are also obtained, which perhaps deserve to be included in standard compilations, since they involve classical polynomials such as the Jacobi polynomials and other 'named' polynomials.

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1. Introduction

Recently certain Diophantine conjectures have been proffered [1, 2] and proven [3] (see also [4]). This entailed the identification of certain classes of orthogonal polynomials $p_n^{(\nu)}(x)$, of degree n in the variable x and depending (also polynomially) on a parameter ν, which feature zeros given by simple formulae involving integers when the parameter ν takes appropriate integer values. In the present paper we report additional findings of this kind, involving known classes of orthogonal polynomials as well as new ones and including (and extending) certain of the results reported recently by Askey [5] and Holtz [6] in connection with a remarkable Diophantine finding presented over a century and a half ago by Sylvester [7]. Our results consist in the identification of classes, defined by three-term recursion relations (see (1)), of orthogonal polynomials some of which—of arbitrary degree n—feature zeros given by neat formulae involving integers, or equivalently in the identification of 'remarkable' tridiagonal matrices—of arbitrary order n, see (3)—whose eigenvalues are likewise given by neat formulae involving integers. Another finding—which is instrumental to get our Diophantine results, but seems of interest in its own right (indeed, might possibly be deemed the most interesting finding of this paper)—identifies classes of orthogonal polynomials, defined by three-term recursion relations and depending on a parameter ν (see (1)), which moreover also satisfy a second recursion involving that parameter (see (6)) and possibly as well some remarkable factorization properties. These findings are presented in section 2 and proven in section 4, while several examples are exhibited in section 3. Some remarks about continuations of these investigations are outlined in section 5. Certain developments whose detailed report in the body of the paper might disrupt the flow of the presentation are confined to four appendices, the last of which reviews tersely the original approach that led us to our findings, by applying it to two new cases and reporting the corresponding results (including the exhibition of some Diophantine formulae).

2. Main results

In this section 2 we present our main results.

Let the class of monic polynomials $p_n^{(\nu)}(x)$, of degree n in the variable x and depending on the parameter ν, be defined by the three-term recursion relation

$$p_{n+1}^{(\nu)}(x) = \left(x + a_n^{(\nu)}\right) p_n^{(\nu)}(x) + b_n^{(\nu)} p_{n-1}^{(\nu)}(x) \tag{1a}$$

with the 'initial' assignment

$$p_{-1}^{(\nu)}(x) = 0, \qquad p_0^{(\nu)}(x) = 1, \tag{1b}$$

clearly entailing

$$p_1^{(\nu)}(x) = x + a_0^{(\nu)}, \qquad p_2^{(\nu)}(x) = \left(x + a_1^{(\nu)}\right)\left(x + a_0^{(\nu)}\right) + b_1^{(\nu)} \tag{1c}$$

and so on.

Notation. Hereafter the index n is a nonnegative integer (but some of the formulae written below might make little sense for $n = 0$, requiring a—generally quite obvious—special interpretation), and $a_n^{(\nu)}, b_n^{(\nu)}$ are functions of this index n and of the parameter ν. These functions are hereafter assumed to be independent of the variable x; although a linear dependence of $a_n^{(\nu)}$ on x and a quadratic dependence of $b_n^{(\nu)}$ on x would not spoil the polynomial character (of degree n) of $p_n^{(\nu)}(x)$. They might also depend on other parameters besides ν (see below); but ν plays a special role, because in the following we shall mainly focus on special values of this parameter (generally simply related to the index n).

Remark 2.1. The polynomials $p_n^{(\nu)}(x)$ are generally orthogonal ('Favard theorem' [8, 10]) but this feature plays no role in the following.

Remark 2.2. The (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ defined by the three-term recursion relation (1) are related to tridiagonal matrices via the well-known formula

$$p_n^{(\nu)}(x) = \det[x - M^{(\nu)}] \tag{2}$$

with the tridiagonal $n \times n$ matrix $M^{(\nu)}$ defined componentwise as follows:

$$M_{m,m+1}^{(\nu)} = \frac{b_m^{(\nu)}}{c_m^{(\nu)}}, \qquad m = 1, \ldots, n-1, \tag{3a}$$

$$M_{m,m}^{(\nu)} = -a_{m-1}^{(\nu)}, \qquad m = 1, \ldots, n, \tag{3b}$$

$$M_{m,m-1}^{(\nu)} = -c_{m-1}^{(\nu)}, \qquad m = 2, \ldots, n \tag{3c}$$

with all other elements vanishing. Here the $n-1$ quantities $c_m^{(\nu)}, m=1,\ldots,n-1$ are arbitrary (of course nonvanishing, $c_m^{(\nu)}\neq 0$, see (3*a*)). These formulae entail that the n zeros of the polynomial $p_n^{(\nu)}(x)$ defined by the three-term recursion relation (1) coincide with the n eigenvalues of the tridiagonal $n\times n$ matrix $M^{(\nu)}$, see (3).

Hence the Diophantine findings reported below, identifying polynomials belonging to orthogonal families that feature zeros given by neat formulae involving integers, might as well be reformulated as identifying tridiagonal matrices that are remarkable inasmuch as they feature eigenvalues given by neat formulae involving integers.

2.1. Second recursion relation

We now report a result concerning the (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ defined by the three-term recursion relations (1). This finding is instrumental to obtain the Diophantine results detailed in the following, but—as already mentioned above—it seems of interest in itself.

Proposition 2.3. *Assume that the quantities $A_n^{(\nu)}$ and $\alpha^{(\nu)}$ satisfy the nonlinear recursion relation*

$$\left[A_{n-1}^{(\nu)}-A_{n-1}^{(\nu-1)}\right]\left[A_n^{(\nu)}-A_{n-1}^{(\nu-1)}+\alpha^{(\nu)}\right] = \left[A_{n-1}^{(\nu-1)}-A_{n-1}^{(\nu-2)}\right]\left[A_{n-1}^{(\nu-1)}-A_{n-2}^{(\nu-2)}+\alpha^{(\nu-1)}\right] \tag{4a}$$

with the boundary condition

$$A_0^{(\nu)}=A \tag{4b}$$

where A is an arbitrary constant (independent of ν), and that the coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ are defined in terms of these quantities by the following formulae:

$$a_n^{(\nu)}=A_{n+1}^{(\nu)}-A_n^{(\nu)}, \tag{5a}$$

$$b_n^{(\nu)}=\left[A_n^{(\nu)}-A_n^{(\nu-1)}\right]\left[A_n^{(\nu)}-A_{n-1}^{(\nu-1)}+\alpha^{(\nu)}\right]. \tag{5b}$$

Then the polynomials $p_n^{(\nu)}(x)$ identified by the recursion relation (1) satisfy the following additional recursion relation (involving a shift both in the order n of the polynomials and in the parameter ν):

$$p_n^{(\nu)}(x)=p_n^{(\nu-1)}(x)+g_n^{(\nu)}p_{n-1}^{(\nu-1)}(x) \tag{6}$$

with

$$g_n^{(\nu)}=A_n^{(\nu)}-A_n^{(\nu-1)}. \tag{7}$$

A more general version of this proposition 2.3 can be formulated, but since we did not (yet) find any interesting application of it we relegate it to appendix A.

It is unlikely that it will be possible to find the general solution of the nonlinear relations (4*a*) with (or possibly without) (4*b*). But nontrivial classes of quantities $A_n^{(\nu)}$ and $\alpha^{(\nu)}$ satisfying the nonlinear relations (4) are provided in section 3—as well as the corresponding coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ (see (5)) and $g_n^{(\nu)}$ (see (7)) defining, via the recursion relations (1), families of (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ satisfying—as entailed by this proposition 2.3—also the second class of recursion relations (6).

Moreover, in appendix B we report several relations implied, by the nonlinear equations (4), for the coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$ and $g_n^{(\nu)}$, including some formulae used in section 4 to prove this proposition 2.3 as well as propositions 2.4 and 2.8 presented below.

2.2. Factorization

Proposition 2.4. *If the (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ are defined by the recursion relation (1) and the coefficients $b_n^{(\nu)}$ satisfy the relation*

$$b_n^{(n)} = 0, \tag{8}$$

entailing that, for $\nu = n$, the recursion relation (1a) reads

$$p_{n+1}^{(n)}(x) = \left(x + a_n^{(n)}\right) p_n^{(n)}(x), \tag{9}$$

then there holds the factorization

$$p_n^{(m)}(x) = \tilde{p}_{n-m}^{(-m)}(x)\, p_m^{(m)}(x), \qquad m = 0, 1, \ldots, n \tag{10}$$

with the 'complementary' polynomials $\tilde{p}_n^{(-m)}(x)$ (of course of degree n) defined by the following three-term recursion relation analogous (but not identical) to (1):

$$\tilde{p}_{n+1}^{(-m)}(x) = \left(x + a_{n+m}^{(m)}\right)\tilde{p}_n^{(-m)}(x) + b_{n+m}^{(m)}\,\tilde{p}_{n-1}^{(-m)}(x), \tag{11a}$$

$$\tilde{p}_{-1}^{(-m)}(x) = 0, \qquad \tilde{p}_0^{(-m)}(x) = 1, \tag{11b}$$

entailing

$$\tilde{p}_1^{(-m)}(x) = x + a_m^{(m)}, \tag{11c}$$

$$\begin{aligned}\tilde{p}_2^{(-m)}(x) &= \left(x + a_{m+1}^{(m)}\right)\left(x + a_m^{(m)}\right) + b_{m+1}^{(m)} \\ &= \left(x - x_m^{(+)}\right)\left(x - x_m^{(-)}\right)\end{aligned} \tag{11d}$$

with

$$x_m^{(\pm)} = \tfrac{1}{2}\left\{-a_m^{(m)} - a_{m+1}^{(m)} \pm \left[\left(a_m^{(m)} - a_{m+1}^{(m)}\right)^2 - 4b_{m+1}^{(m)}\right]^{1/2}\right\} \tag{11e}$$

and so on.

The following two results are immediate consequences of proposition 2.4.

Corollary 2.5. *If (8) holds—entailing (9) and (10) with (11)—the polynomial $p_n^{(n-1)}(x)$ has the zero $-a_{n-1}^{(n-1)}$,*

$$p_n^{(n-1)}\left(-a_{n-1}^{(n-1)}\right) = 0, \tag{12a}$$

and the polynomial $p_n^{(n-2)}(x)$ has the two zeros $x_{n-2}^{(\pm)}$, see (11e),

$$p_n^{(n-2)}\left(x_{n-2}^{(\pm)}\right) = 0. \tag{12b}$$

The first of these results is a trivial consequence of (9); the second is evident from (10) and (11). Note moreover that from the factorization formula (10) one can likewise find explicitly three zeros of $p_n^{(n-3)}(x)$ and four zeros of $p_n^{(n-4)}(x)$, by evaluating from (11) $\tilde{p}_3^{(-m)}(x)$ and $\tilde{p}_4^{(-m)}(x)$ and by taking advantage of the explicit solvability of algebraic equations of degree 3 and 4.

Corollary 2.6. *If (8) holds—entailing (9) and (10) with (11)—and moreover the quantities $a_n^{(m)}$ and $b_n^{(m)}$ satisfy the symmetry properties*

$$a_{n-m}^{(-m)} = a_n^{(m)}, \qquad b_{n-m}^{(-m)} = b_n^{(m)} \tag{13}$$

then

$$\tilde{p}_n^{(m)}(x) = p_n^{(m)}(x) \tag{14}$$

entailing that factorization (10) takes the neat form

$$p_n^{(m)}(x) = p_{n-m}^{(-m)}(x)\,p_m^{(m)}(x), \qquad m = 0, 1, \ldots, n. \tag{15}$$

Note that these results about factorization only hold for polynomials $p_n^{(\nu)}(x)$ with the parameter ν taking integer values.

The following remark is relevant when both propositions 2.3 and 2.4 hold.

Remark 2.7. As implied by (5*b*), condition (8) can be enforced via the assignment

$$\alpha^{(\nu)} = A_{\nu-1}^{(\nu-1)} - A_\nu^{(\nu)} \tag{16}$$

entailing that the nonlinear recursion relation (5*a*) reads

$$\begin{aligned}&\left[A_{n-1}^{(\nu)} - A_{n-1}^{(\nu-1)}\right]\left[A_n^{(\nu)} - A_{n-1}^{(\nu-1)} + A_{\nu-1}^{(\nu-1)} - A_\nu^{(\nu)}\right]\\ &\quad = \left[A_{n-1}^{(\nu-1)} - A_{n-1}^{(\nu-2)}\right]\left[A_{n-1}^{(\nu-1)} - A_{n-2}^{(\nu-2)} + A_{\nu-2}^{(\nu-2)} - A_{\nu-1}^{(\nu-1)}\right].\end{aligned} \tag{17}$$

2.3. Diophantine findings

Proposition 2.8. *If the (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ are defined by the three-term recursion relations (1) with coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ satisfying the requirements sufficient for the validity of both propositions 2.3 and proposition 2.4 (namely (5) with (4) and (8) or just with (17)), then*

$$p_n^{(n)}(x) = \prod_{m=1}^{n} (x - x_m), \tag{18a}$$

with the following (n-independent) expression of the n zeros x_m:

$$x_m = A_{m-1}^{(m-1)} - A_m^{(m)} \tag{18b}$$

or equivalently (see (5a) and (7))

$$x_m = -\left(a_{m-1}^{(m-1)} + g_m^{(m)}\right). \tag{18c}$$

The following results are immediate consequences of this proposition 2.8 and of corollary 2.5

Corollary 2.9. *If proposition 2.8 holds, then also the polynomials $p_n^{(n-1)}(x)$ and $p_n^{(n-2)}(x)$ (in addition to $p_n^{(n)}(x)$, see (18)) can be written explicitly as*

$$p_n^{(n-1)}(x) = \left(x + a_{n-1}^{(n-1)}\right) \prod_{m=1}^{n-1} (x - x_m), \tag{19a}$$

$$p_n^{(n-2)}(x) = \left[\left(x + a_{n-1}^{(n-2)}\right)\left(x + a_{n-2}^{(n-2)}\right) + b_{n-1}^{(n-2)}\right] \prod_{m=1}^{n-2} (x - x_m). \tag{19b}$$

The Diophantine character of these findings—as indicated in the title of this section 2.3—emerges from the explicitly factorized expressions—generally involving integers—of the polynomial $p_n^{(n)}(x)$, see (18), and of the polynomials $p_n^{(n-1)}(x)$ and $p_n^{(n-2)}(x)$, see (19) and the examples below. Analogously explicit results can clearly be written for the polynomials $p_n^{(n-3)}(x)$ and $p_n^{(n-4)}(x)$, see the last part of corollary 2.5.

3. Examples

In this section 3, we report some assignments of the quantities $A_n^{(\nu)} a^{(\nu)}$—hence correspondingly of the coefficients $a_n^{(\nu)}, b_n^{(\nu)}$ and $g_n^{(\nu)}$, see (5) and (7)—guaranteeing the validity of proposition 2.3, and often as well of the other results reported in the preceding section 2; and whenever appropriate we tersely discuss the corresponding polynomials, which are often related to known (named) ones. But before delving into the exhibition of various examples, let us report the following, rather obvious

Remark 3.1. If a set of coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$ and $g_n^{(\nu)}$ satisfies the requirements sufficient to guarantee the validity of the results reported in section 2, the following extension of it,

$$\check{a}_n^{(\nu)} = \gamma a_n^{(\nu)} + \alpha, \qquad \check{b}_n^{(\nu)} = \gamma^2 b_n^{(\nu)}, \qquad \check{g}_n^{(\nu)} = \gamma g_n^{(\nu)}, \tag{20a}$$

with α and γ two arbitrary parameters, also satisfies the same conditions; this extension being clearly related to the following transformation of the corresponding polynomials:

$$\check{p}_n^{(\nu)}(x) = \gamma^n p_n^{(\nu)}\left(\frac{x+\alpha}{\gamma}\right). \tag{20b}$$

Note that the polynomials $\check{p}_n^{(\nu)}(x)$ are as well monic.

In the examples presented below, we generally refrain from reducing the number of free parameters by exploiting systematically this remark 3.1, since this might obfuscate rather than highlight the transparency of our findings. The diligent reader is welcome to verify the consistency of all the findings reported below with the validity of this remark 3.1.

3.1. Polynomial solution of (4)

The following assignment satisfies the nonlinear conditions (4):

$$A_n^{(\nu)} = k_0 + k_1 n + k_2 n^2 + k_3 n^3 + \left(k_4 n - \tfrac{3}{2} k_3 n^2\right)\nu \tag{21a}$$

with

$$\alpha^{(\nu)} = -k_1 + k_2 + \tfrac{1}{2}k_3 + k_4 + k_5 - \left(2k_2 + \tfrac{3}{2}k_3 + 2k_4\right)\nu + \tfrac{3}{2}k_3\nu^2, \tag{21b}$$

$$A = k_0. \tag{21c}$$

Here the five parameters k_j are arbitrary.

The corresponding expressions of the coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$ and $g_n^{(\nu)}$ read

$$a_n^{(\nu)} = k_1 + k_2 + k_3 + \left(-\tfrac{3}{2}k_3 + k_4\right)\nu + [2k_2 + 3k_3(1-\nu)]n + 3k_3 n^2, \tag{21d}$$

$$b_n^{(\nu)} = -\tfrac{1}{4}n(3k_3 n - 2k_4)[2k_5 + 2(2k_2 + k_4)(n-\nu) + 3k_3(n-\nu)^2], \tag{21e}$$

$$g_n^{(\nu)} = -\tfrac{1}{2}n(3k_3 n - 2k_4). \tag{21f}$$

Hereafter we identify our polynomials $p_n^{(\nu)}(x)$ belonging to this class—hence satisfying proposition 2.3—as $p_n^{(\nu)}(x; k_1, k_2, k_3, k_4, k_5)$.

Remark 3.2. It is plain (see (21e) and (8)) that the subclass $p_n^{(\nu)}(x; k_1, k_2, k_3, k_4, 0)$ of these polynomials also satisfies propositions 2.4 and 2.8, entailing the factorization

$$p_n^{(n)}(x; k_1, k_2, k_3, k_4, 0) = \prod_{m=1}^{n}(x - x_m) \tag{22a}$$

with

$$x_m = \alpha^{(m)} = -k_1 + k_2 + \tfrac{1}{2}k_3 + k_4 - \big(2k_2 + \tfrac{3}{2}k_3 + 2k_4\big)m + \tfrac{3}{2}k_3 m^2, \tag{22b}$$

as well as

$$p_n^{(n-1)}(x; k_1, k_2, k_3, k_4, 0) = (x - \hat{x}_n) \prod_{m=1}^{n-1} (x - x_m) \tag{23a}$$

with

$$\hat{x}_n = -k_1 + k_2 + \tfrac{1}{2}k_3 + k_4 - \big(2k_2 + \tfrac{3}{2}k_3 + k_4\big)n, \tag{23b}$$

and

$$p_n^{(n-2)}(x; k_1, k_2, k_3, k_4, 0) = \big(x - \hat{x}_n^{(+)}\big)(x - \hat{x}_n^{(-)}) \prod_{m=1}^{n-2} (x - x_m) \tag{24a}$$

with

$$\hat{x}_n^{(\pm)} = -k_1 + 2(k_2 + k_3 + k_4) - (2k_2 + 3k_3 + k_4)n \pm \tfrac{1}{2}\sqrt{z_n}) \tag{25}$$

where

$$z_n = (2k_2 + 3k_3 + 2k_4)^2 - 2(3k_3 + 2k_4)(2k_2 + 3k_3 + k_4)n + 6k_3(2k_2 + 3k_3 + k_4)n^2. \tag{26}$$

Obviously there are many special cases in which z_n becomes a perfect square, for instance

$$k_3 = 0, \qquad k_4 = -2k_2, \qquad z_n = (2k_2)^2 \tag{27a}$$

yielding

$$\hat{x}_n^{(+)} = -k_1 - k_2, \qquad \hat{x}_n^{(-)} = -k_1 - 3k_2; \tag{27b}$$

$$k_2 = 0, \qquad k_4 = -\tfrac{3}{2}k_3, \qquad z_n = (3k_3 n)^2 \tag{28a}$$

yielding

$$\hat{x}_n^{(+)} = -k_1 - k_3, \qquad \hat{x}_n^{(-)} = -k_1 - k_3 - 3k_3 n; \tag{28b}$$

$$k_2 = -\tfrac{1}{2}k, \qquad k_3 = \tfrac{1}{3}, \qquad k_4 = -\tfrac{1}{2} + k, \qquad z_n = (n-k)^2 \tag{29a}$$

yielding

$$\hat{x}_n^{(+)} = -k_1 - \frac{1}{3} + \frac{1}{2}k, \qquad \hat{x}_n^{(-)} = -k_1 - \frac{1}{3} + \frac{3}{2}k - n; \tag{29b}$$

$$k_2 = \frac{k(2k-3)}{2(2k-1)}, \qquad k_3 = \frac{1}{3(2k-1)}, \qquad k_4 = \frac{1}{2}, \qquad z_n = (n-k)^2 \tag{30a}$$

yielding

$$\hat{x}_n^{(\pm)} = -k_1 + \frac{(2 \pm 1)}{2}k - \frac{1}{3(2k-1)} + \left(\frac{1 \pm 1}{2} - k\right)n. \tag{30b}$$

There moreover holds factorization (10) and, for the subclass of polynomials $p_n^{(\nu)}(x; k_1, k_2, k_3, -k_2, 0)$, factorization (15),

$$p_n^{(m)}(x; k_1, k_2, k_3, -k_2, 0) = p_{n-m}^{(-m)}(x; k_1, k_2, k_3, -k_2, 0)\, p_m^{(m)}(x; k_1, k_2, k_3, -k_2, 0), \quad m = 0, 1, \ldots, n. \tag{31}$$

In the following subsections we report a few specific examples involving 'named' polynomials; examples involving other named polynomials are in hand but they are not presented here to avoid overburdening this paper. For an additional example, including the exhibition of Diophantine matrices, see appendix D.

3.1.1. Laguerre polynomials. The 'normalized Laguerre polynomials' $\pounds_n^{(\alpha)}(x)$, related to the usual generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ by the formula

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}\pounds_n^{(\alpha)}(x), \tag{32}$$

are the following special case of the polynomials $p_n^{(\nu)}(x; k_1, k_2, k_3, k_4, k_5)$,

$$\pounds_n^{(\alpha)}(x) = p_n^{(-\alpha)}(x; 0, -1, 0, 1, 0) \tag{33}$$

as seen by comparing the recursion relation (1.11.4) of http://aw.twi.tudelft.nl/~koekoek/askey/ch1/par11/par11.html [9] with our recursion relation (1) with (21*d*) and (21*e*). Note that it was actually unnecessary to set $k_5 = 0$ on the right-hand side of this formula, (33), since—as can be easily seen—any value of k_5 yields in this case the same outcome; by setting $k_5 = 0$ we made it evident that these polynomials satisfy not only proposition 2.3, but also propositions 2.4 and 2.8. Hence the normalized Laguerre polynomials $\pounds_n^{(\alpha)}(x)$ satisfy the second recursion relation (see (7))

$$\pounds_n^{(\alpha)}(x) = \pounds_n^{(\alpha+1)}(x) + n\pounds_{n-1}^{(\alpha+1)}, \tag{34a}$$

and correspondingly the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ satisfy the (well-known) second recursion relation

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}. \tag{34b}$$

Likewise the normalized Laguerre polynomials satisfy the factorization

$$\begin{aligned}\pounds_n^{(-m)}(x) &= \pounds_{n-m}^{(m)}(x)\pounds_m^{(-m)}(x)\\ &= x^m\pounds_{n-m}^{(m)}(x), \qquad m = 0, 1, \ldots, n\end{aligned} \tag{35a}$$

entailing for the generalized Laguerre polynomials the formula

$$L_n^{(-m)}(x) = \frac{m!(n-m)!}{n!}x^m L_{n-m}^{(m)}(x), \qquad m = 0, 1, \ldots, n. \tag{35b}$$

And the previous findings entail that the generalized Laguerre polynomials $L_n^{(-m)}(x)$ satisfy the following properties (displaying the Diophantine character of their zeros):

$$L_n^{(-n)}(x) = \frac{(-1)^n}{n!}x^n, \tag{36a}$$

$$L_n^{(-n+1))}(x) = \frac{(-1)^n}{n!}x^{n-1}(x-n), \tag{36b}$$

$$L_n^{(-n+2))}(x) = \frac{(-1)^n}{n!}(x-n-\sqrt{n})(x-n+\sqrt{n})x^{n-2}, \tag{36c}$$

implying, for instance, the additional Diophantine finding

$$L_{n^2}^{(-n^2+2))}(x) = \frac{(-1)^{n^2}}{n^2!}[x-n(n+1)][x-n(n-1)]x^{n^2-2}. \tag{36d}$$

Some (but not all) of these formulae are reported in the standard compilations [9–12].

3.1.2. Meixner polynomials. The 'normalized Meixner polynomials' $\tilde{M}_n(x; \beta, c)$, related to the usual Meixner polynomials $M_n^{(\alpha)}(x; \beta, c)$ by the formula

$$M_n(x; \beta, c) = \frac{1}{(\beta)_n}\left(\frac{c-1}{c}\right)^n \tilde{M}_n(x; \beta, c), \tag{37}$$

are the following special case of the polynomials $p_n^{(\nu)}(x; k_1, k_2, k_3, k_4, k_5)$,

$$\tilde{M}_n(x; \beta, c) = p_n^{(-\beta)}\left(x; -\frac{1}{2}\frac{c+1}{c-1}, \frac{1}{2}\frac{c+1}{c-1}, 0, -\frac{c}{c-1}, -\frac{1}{c-1}\right), \tag{38}$$

as seen by comparing the recursion relation (1.9.4) of http://aw.twi.tudelft.nl/~koekoek/askey/ch1/par9/par9.html [9] with our recursion relation (1) (with (21d) and (21e)). Note that in this case the condition $k_5 = 0$ cannot be enforced (except for $c = \infty$), so these polynomials satisfy proposition 2.3 but not propositions 2.4 and 2.8. Hence the normalized Meixner polynomials $\tilde{M}_n(x; \beta, c)$ satisfy the second recursion relation (see (7))

$$\tilde{M}_n(x; \beta, c) = \tilde{M}_n(x; \beta+1, c) - \frac{c}{c-1} n \tilde{M}_{n-1}(x; \beta+1, c), \tag{39a}$$

and correspondingly the usual Meixner polynomials $M_n(x; \beta, c)$ satisfy the second recursion relation

$$\beta M_n(x; \beta, c) = (\beta+n) M_n(x; \beta+1, c) - n(\beta+n) M_{n-1}(x; \beta+1, c), \tag{39b}$$

which is not reported in the standard compilations [9–12].

3.1.3. Continuous Dual Hahn (CDH) polynomials. The 'normalized CDH polynomials' $\tilde{S}_n(x; \alpha, \beta, \gamma)$, related to the usual CDH polynomials $S_n(x; \alpha, \beta, \gamma)$ by the formula

$$S_n(x; \alpha, \beta, \gamma) = (-1)^n \tilde{S}_n(x; \alpha, \beta, \gamma), \tag{40}$$

are the following special case of the polynomials $p_n^{(\nu)}(x; k_1, k_2, k_3, k_4, k_5)$,

$$\tilde{S}_n(x; \alpha, \beta, \gamma) = p_n^{(-\alpha)}\big(x; \beta+\gamma-\beta\gamma-\tfrac{5}{6}, -\beta-\gamma+\tfrac{3}{2}, -\tfrac{2}{3}, \beta+\gamma-1, \beta+\gamma-1-\beta\gamma\big) \tag{41}$$

as seen by comparing the recursion relation (1.3.5) of http://aw.twi.tudelft.nl/~koekoek/askey/ch1/par3/par3.html [9] with our recursion relation (1) (with (21d) and (21e)); note that, since these polynomials are completely symmetrical under the exchange of their three parameters α, β, γ, two other identifications could of course have been made. Hence the CDH polynomials $S_n(x; \alpha, \beta, \gamma)$ satisfy the second recursion relation (see (7))

$$S_n(x; \alpha, \beta, \gamma) = S_n(x; \alpha+1, \beta, \gamma) - n(n+\beta+\gamma-1) S_{n-1}(x; \alpha+1, \beta, \gamma), \tag{42}$$

which is not reported in the standard compilations [9–12]. It is moreover clear that for the following subclass of these polynomials:

$$S_n(x; \alpha, \beta, 1) = (-1)^n p_n^{(\alpha)}\big(x; \tfrac{1}{6}, \tfrac{1}{2}-\beta, -\tfrac{2}{3}, \beta, 0\big) \tag{43}$$

there holds factorization (10) (although not factorization (15)); hence the following explicit expressions:

$$S_n(x; -n, \beta, 1) = (-1)^n \prod_{m=1}^{n} (x+m^2), \tag{44}$$

$$S_n(x; -n+1, \beta, 1) = (-1)^n (x-\beta n) \prod_{m=1}^{n-1} (x+m^2), \tag{45}$$

$$S_n(x; -n+2, \beta, 1) = (-1)^n [x^2 + (-2n+1-2\beta n)x + \beta(\beta+1)n(n-1)] \prod_{m=1}^{n-2} (x+m^2), \tag{46}$$

entailing

$$S_n(x; -n+2, -1, 1) = (-1)^n x(x+1)\prod_{m=1}^{n-2}(x+m^2), \tag{47a}$$

$$S_n(x; -n+2, 0, 1) = (-1)^n x(x+1-2n)\prod_{m=1}^{n-2}(x+m^2), \tag{47b}$$

$$S_n\left(x; -n+2, \frac{1}{2}(1\pm\sqrt{5}), 1\right) = (-1)^n(x+1-n)[x-(2\pm\sqrt{5})n]\prod_{m=1}^{n-2}(x+m^2), \tag{47c}$$

whose Diophantine character (in terms of the zeros of these polynomials) is plain. These formulae are not reported in the standard compilations [9–12].

3.1.4. Askey's B polynomials. Let us introduce the following modified version of the polynomials $B_n(x; a, \mu)$ introduced by Askey [5], via the position

$$\hat{B}_n(x; a, \mu) = B_n(x + a\mu; a, \mu). \tag{48}$$

The motivation for modifying in this manner Askey's B-polynomials will be clear below. Let us moreover emphasize that we allow the parameter μ to be an arbitrary number (while it was restricted to be an integer in [5]).

It is easily seen that these polynomials $\hat{B}_n(x; a, \mu, \beta)$ are a subclass of our polynomials $p_n^{(\nu)}(x; k_1, k_2, k_3, k_4, k_5)$:

$$\hat{B}_n(x; a, \mu) = p_n^{(\mu+1+k)}\left[x; -a(1+k) + \tfrac{1}{2}, -\tfrac{1}{2}, 0, a, k(a-1)\right]. \tag{49a}$$

It is moreover plain that the parameter k appearing on the right-hand side of this formula plays no role, hence hereafter we set it to zero:

$$\hat{B}_n(x; a, \mu) = p_n^{(\mu+1)}\left(x; -a + \tfrac{1}{2}, -\tfrac{1}{2}, 0, a, 0\right). \tag{49b}$$

It is thereby clear that these polynomials satisfy condition (8) hence satisfy proposition 2.4 (see remark 3.2), in addition of course to propositions 2.3 and 2.8 (while clearly factorization (15) only holds for $a = 1/2$).

Hence these polynomials satisfy the second recurrence relation,

$$\hat{B}_n(x; a, \mu) = \hat{B}_n(x; a, \mu - 1) + an\hat{B}_{n-1}(x; a, \mu - 1), \tag{50}$$

and there holds for their subclass with $a = 1/2$ the factorization

$$\hat{B}_n\left(x; \tfrac{1}{2}, m-1\right) = \hat{B}_{n-m}\left(x; \tfrac{1}{2}, -m-1\right)\right) \cdot \hat{B}_m\left(x; \tfrac{1}{2}, m-1\right), \qquad m = 1, 2, \ldots, n. \tag{51}$$

Let us emphasize that, due to the definition (48), a shift in the parameter μ of the polynomials $\hat{B}_n(x; a, \mu)$ also entails a shift in the variable x for the polynomials $B_n(x; a, \mu)$.

Moreover for these polynomials there hold the Diophantine factorizations

$$\hat{B}_n(x; a, n-1) = \prod_{m=1}^{n}(x - x_m), \tag{52a}$$

$$x_m = (2a-1)(1-m); \tag{52b}$$

$$\hat{B}_n(x; a, n-2) = (x - \hat{x}_n) \prod_{m=1}^{n-1} (x - x_m), \tag{53a}$$

$$\hat{x}_n = a - (1-a)(1-n); \tag{53b}$$

$$\hat{B}_n(x; a, n-3) = \big(x - \hat{x}_n^{(+)}\big)\big(x - \hat{x}_n^{(-)}\big) \prod_{m=1}^{n-2} (x - x_m), \tag{54a}$$

$$\hat{x}_n^{(\pm)} = 3\left(a - \frac{1}{2}\right) + (1-a)n \pm \frac{1}{2}\sqrt{(1-2a)^2 + 4a(1-a)n}. \tag{54b}$$

Hence, in addition to the (already known [5]) simple cases ($a = 0, a = 1$) when the original three-term relation becomes a two-term relation, additional Diophantine (i.e., integer respectively rational) zeros occur, for instance, for $n = m^2, a = 1/2$ entailing $\hat{x}_n^{(\pm)} = m(m \pm 1)/2$ respectively for $n = m^2, a = (4m^2 - 1)/[2(2m^2 - 1)]$ entailing $\hat{x}_n^{(\pm)} = m(5m \pm 1)/[2(2m^2 - 1)]$.

3.2. Rational solution of (4)

The following assignment satisfies the nonlinear conditions (4):

$$A_n^{(\nu)} = \frac{n\left(c_0c_1 + \left(c_1 - c_2 + c_0c_3 + 3c_0^2c_4\right)\nu + (c_2 + c_3\nu)n + c_4(2c_0 - 2\nu + n)n^2\right)}{(c_0 + 2n - \nu)} \tag{55a}$$

with

$$\alpha^{(\nu)} = -c_1 + c_3 + c_4(1 + 3c_0) - (2c_3 + 3c_4(1 + 2c_0))\nu + 3c_4\nu^2, \tag{55b}$$

$$A = 0. \tag{55c}$$

Here the five parameters c_j are arbitrary.

The corresponding expressions of the coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$ and $g_n^{(\nu)}$ read

$$a_n^{(\nu)} = \frac{a(n,\nu)}{(c_0 + 2n - \nu)(c_0 + 2 + 2n - \nu)}, \tag{56a}$$

$$\begin{aligned} a(n,\nu) = {} & c_0[c_2 + c_4 + c_0(c_1 + 2c_4)] - (1 + c_0)\{c_2 + c_4 - c_0[c_3 + 3c_4(c_0 - 1)]\}\nu \\ & - [c_0(c_3 + 3c_0c_4) + c_1 - c_2 + c_3 - 2c_4]\nu^2 + (1 + c_0)[c_2 + c_4(1 + 3c_0)]n \\ & + 2[c_0(c_3 - 6c_4) - c_2 + c_3 - 4c_4]n\nu - 2(c_3 - 3c_4)n\nu^2 \\ & + 2\big[c_2 + c_4\big(4 + 9c_0 + 3c_0^2\big)\big]n^2 + 2[c_3 - 3c_4(3 + 2c_0)]n^2\nu + 6c_4n^2\nu^2 \\ & + 12(1 + c_0)c_4n^3 - 12c_4n^3\nu + 6c_4n^4; \end{aligned} \tag{56b}$$

$$b_n^{(\nu)} = \frac{n(n-\nu)(c_0 + n)(c_0 + n - \nu)\tilde{b}(n,\nu)\hat{b}(n,\nu)}{(c_0 + 2n - \nu)^2(c_0 + 1 + 2n - \nu)(c_0 - 1 + 2n - \nu)}, \tag{57a}$$

$$\tilde{b}(n,\nu) = c_0(c_3 + 3c_0c_4) + 2c_1 - c_2 + (2c_3 + 3c_0c_4)n - 3c_4n^2, \tag{57b}$$

$$\hat{b}(n,\nu) = c_0(c_3 + 3c_0c_4) - 2c_1 + c_2 + 3c_4\nu^2 + (2c_3 + 9c_0c_4)(n - \nu) - 6c_4\nu n + 3c_4n^2, \tag{57c}$$

$$g_n^{(\nu)} = \frac{n(c_0 + n)(c_0(c_3 + 3c_0c_4) + 2c_1 - c_2 + (2c_3 + 3c_0c_4)n - 3c_4n^2)}{(c_0 + 2n - \nu)(c_0 + 1 + 2n - \nu)}. \tag{58}$$

Hereafter we identify our polynomials $p_n^{(\nu)}(x)$ belonging to this class—hence satisfying proposition 2.3—as $p_n^{(\nu)}(x; c_0, c_1, c_2, c_3, c_4)$. Of course they should not be confused with the polynomials introduced in the preceding section 3.1.

It is plain (see (57*a*) and (8)) that these polynomials also satisfy propositions 2.4 and 2.8, entailing the factorizations

$$p_n^{(n)}(x; c_0, c_1, c_2, c_3, c_4) = \prod_{m=1}^{n}(x - x_m), \tag{59a}$$

$$x_m = \alpha^{(m)} = -c_1 + c_3 + c_4(1 + 3c_0) - [2c_3 + 3c_4(1 + 2c_0)]m + 3c_4 m^2; \tag{59b}$$

$$p_n^{(n-1)}(x; c_0, c_1, c_2, c_3, c_4) = (x - \hat{x}_n)\prod_{m=1}^{n-1}(x - x_m), \tag{60a}$$

$$\hat{x}_n = -\frac{(1 + c_0)[c_1 - c_3 - c_4(1 + 3c_0)] + [c_0(c_3 + 3c_4(2 + c_0)) - c_1 + c_2 + c_3 + 2c_4]n}{1 + c_0 + n}. \tag{60b}$$

Three interesting cases that deserve to be highlighted read as follows:

$$p_n^{(n-1)}(x; -1, c_1, c_2, c_3, c_4) = (x - \hat{x}_n)\prod_{m=1}^{n-1}(x - x_m), \tag{61a}$$

$$\hat{x}_n = c_1 - c_2 + c_4, \tag{61b}$$

$$x_m = -c_1 + c_3 - 2c_4 - (2c_3 - 3c_4)m + 3c_4 m^2; \tag{61c}$$

$$p_n^{(n-1)}(x; 0, c_2 + c_3 + 2c_4, c_2, c_3, c_4) = (x - \hat{x}_n)\prod_{m=1}^{n-1}(x - x_m), \tag{62a}$$

$$\hat{x}_n = -\frac{c_2 + c_4}{n + 1}, \tag{62b}$$

$$x_m = -(c_2 + c_4 + (2c_3 + 3c_4)m - 3c_4 m^2); \tag{62c}$$

$$p_n^{(n-1)}\left(x; 0, -\frac{1 + 3\sigma}{2}c_4, -c_4 - \rho, -\frac{3}{2}(1 + \sigma)c_4, c_4\right) = (x - \hat{x}_n)\prod_{m=1}^{n-1}(x - x_m), \tag{63a}$$

$$\hat{x}_n = \rho\frac{n}{n + 1}, \tag{63b}$$

$$x_m = 3c_4 m(m + \sigma). \tag{63c}$$

Moreover

$$p_n^{(n-2)}(x; c_0, c_1, c_2, c_3, c_4) = \left(x - \hat{x}_n^{(+)}\right)\left(x - \hat{x}_n^{(-)}\right)\prod_{m=1}^{n-2}(x - x_m). \tag{64}$$

We do not report the (rather complicated) expressions of the two zeros $\hat{x}_n^{(\pm)}$, except in the following special cases:

$$p_n^{(n-2)}(x; -1, c_1, c_2, c_3, c_4) = \left(x - \hat{x}_n^{(+)}\right)\left(x - \hat{x}_n^{(-)}\right)\prod_{m=1}^{n-2}(x - x_m), \tag{65a}$$

$$\hat{x}_n^{(+)} = \frac{-3c_1 + c_2 + 4c_3 - 5c_4 + (c_1 - c_2 - 2c_3 + c_4)n}{n+1}, \tag{65b}$$

$$\hat{x}_n^{(-)} = c_1 - c_2 + c_4, \tag{65c}$$

with the zeros x_m given by (61*c*);

$$p_n^{(n-2)}(x; -2, c_1, c_2, c_3, 0) = \big(x - \hat{x}_n^{(+)}\big)\big(x - \hat{x}_n^{(-)}\big) \prod_{m=1}^{n-2} (x - x_m), \tag{66a}$$

$$\hat{x}_n^{(\pm)} = c_1 - c_2 \pm c_3, \tag{66b}$$

$$x_m = -c_1 + c_3 - 2c_3 m. \tag{66c}$$

The enterprising reader will surely identify several other remarkable cases.

Moreover there holds factorization (10) and, for the subclass of polynomials with

$$c_3 = -3c_0 c_4 \tag{67a}$$

factorization (15),

$$p_n^{(m)}(x; c_0, c_1, c_2, -3c_0c_4, c_4) = p_{n-m}^{(-m)}(x; c_0, c_1, c_2, -3c_0c_4, c_4)\, p_m^{(m)}(x; c_0, c_1, c_2, -3c_0c_4, c_4),$$
$$m = 0, 1, \ldots, n. \tag{67b}$$

3.2.1. Jacobi polynomials. The 'normalized Jacobi polynomials' $\tilde{P}_n^{(\alpha,\beta)}(x)$, related to the usual Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ by the formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)_n}{2^n n!} \tilde{P}_n^{(\alpha,\beta)}(x), \tag{68}$$

are the following special case of the polynomials $p_n^{(\nu)}(x; c_0, c_1, c_2, c_3, c_4)$:

$$\tilde{P}_n^{(\alpha,\beta)}(x) = p_n^{(-\beta)}(x; \alpha, 1, 0, 0, 0), \tag{69}$$

as seen by comparing the recursion relation (1.8.4) of http://aw.twi.tudelft.nl/~koekoek/askey/ch1/par8/par8.html#par1 [9] with our recursion relation (1). Here, and always in the following, additional relations are implied by the well-known symmetry of Jacobi polynomials under the exchange of the two parameters they feature,

$$P_n^{(\alpha,\beta)}(x) = P_n^{(\beta,\alpha)}(-x), \qquad \tilde{P}_n^{(\alpha,\beta)}(x) = \tilde{P}_n^{(\beta,\alpha)}(-x). \tag{70}$$

It is evident that these polynomials (see (69)) satisfy propositions 2.3, 2.4 and 2.8. Hence the normalized Jacobi polynomials $\tilde{P}_n^{(\alpha,\beta)}(x)$ satisfy the second recursion relation (see (7))

$$\tilde{P}_n^{(\alpha,\beta)}(x) = \tilde{P}_n^{(\alpha,\beta+1)}(x) + \frac{2n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} \tilde{P}_n^{(\alpha,\beta+1)}(x), \tag{71a}$$

and correspondingly the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy the (well-known) second recursion relation

$$(2n+\alpha+\beta+1) P_n^{(\alpha,\beta)}(x) = (n+\alpha+\beta+1) P_n^{(\alpha,\beta+1)}(x) + (n+\alpha) P_{n-1}^{(\alpha,\beta+1)}(x). \tag{71b}$$

There holds moreover the following (well-known) Diophantine factorization formula

$$\tilde{P}_n^{(\alpha,-n)}(x) = (x+1)^n \tag{72}$$

as well as (see (60*b*))

$$\tilde{P}_n^{(\alpha,-n+1)}(x) = \left(x + \frac{\alpha+1-n}{\alpha+1+n}\right)(x+1)^{n-1}, \tag{73}$$

$$\tilde{P}_n^{(\alpha,-n+2)}(x) = \left(x - \hat{x}_n^{(+)}\right)\left(x - \hat{x}_n^{(-)}\right)(x+1)^{n-2}, \tag{74a}$$

$$\hat{x}_n^{(\pm)} = \frac{n(n-1) - (\alpha+1)(\alpha+2) \pm 2\sqrt{(\alpha+2)n(n+\alpha+1)}}{n(n+2\alpha+3) + (\alpha+1)(\alpha+2)}. \tag{74b}$$

In particular for $\alpha = -2$

$$\tilde{P}_n^{(-2,-n+2)}(x) = (x-1)^2(x+1)^{n-2} \tag{74c}$$

and for $\alpha = -1$

$$\tilde{P}_n^{(-1,-n+2)}(x) = \left(x - \frac{n-3}{n+1}\right)(x-1)(x+1)^{n-2}. \tag{74d}$$

And clearly there are additional Diophantine zeros whenever $(\alpha+2)n(n+\alpha+1)$ is a perfect square, for instance

$$\tilde{P}_n^{(\alpha,-n+2)}(x) = \left[x - \frac{n-\delta(2\delta+1)}{n+\delta}\right]\left[x - \frac{n^2-(2\delta^2+3\delta+3)n-\delta}{n^2+(3\delta+1)n+\delta(2\delta+1)}\right](x+1)^{n-2}, \tag{74e}$$

$$\alpha = \frac{(\delta^2+2\delta-1)n+2\delta^2}{n-\delta^2}, \tag{74f}$$

$$\tilde{P}_{1+2k(k-1)}^{(\alpha,\beta)}(x) = \left(x - \frac{2k-1}{4k(k-1)+1}\right)^2 (x+1)^{-1+2k(k-1)}, \tag{74g}$$

$$\alpha = -1+2k(k-1), \qquad \beta = -1-2k(k-1), \qquad k = 2, 3, \ldots. \tag{74h}$$

Perhaps (some of) these formulae deserve to be included in the standard compilations.

4. Proofs

In this section, we prove the three propositions reported in the preceding section 2 (also using some formulae of appendix B).

The proof of proposition 2.3 (i.e., of (6)) is by induction. Clearly this relation holds for $n = 1$ (via (1*c*) and (B.1*d*)). Let us assume that it holds up to n, and prove that it then holds for $n+1$. Indeed using (6) on the right-hand side of the recursion relation (1*a*) we get

$$p_{n+1}^{(\nu)}(x) = \left(x + a_n^{(\nu)}\right)\left[p_n^{(\nu-1)}(x) + g_n^{(\nu)} p_{n-1}^{(\nu-1)}(x)\right] + b_n^{(\nu)}\left[p_{n-1}^{(\nu-1)}(x) + g_{n-1}^{(\nu)} p_{n-2}^{(\nu-1)}(x)\right]. \tag{75}$$

We then note that the recursion relation (1*a*) entails the formulae

$$p_{n-1}^{(\nu-1)}(x) = \frac{p_{n+1}^{(\nu-1)}(x) - \left(x + a_n^{(\nu-1)}\right)p_n^{(\nu-1)}(x)}{b_n^{(\nu-1)}}, \tag{76a}$$

$$p_{n-2}^{(\nu-1)}(x) = \frac{\left[b_n^{(\nu-1)} + \left(x + a_{n-1}^{(\nu-1)}\right)\left(x + a_n^{(\nu-1)}\right)\right]p_n^{(\nu-1)}(x)}{b_n^{(\nu-1)} b_{n-1}^{(\nu-1)}} - \frac{\left(x + a_{n-1}^{(\nu-1)}\right)p_{n+1}^{(\nu-1)}(x)}{b_n^{(\nu-1)} b_{n-1}^{(\nu-1)}}. \tag{76b}$$

The second, (76*b*), of these two formulae is of course obtained by replacing n with $n-1$ in the first, (76*a*), and then by using again (76*a*) to eliminate $p_{n-1}^{(\nu-1)}(x)$.

Remark 4.1. This equation, (76*b*), holds for $n \geqslant 2$, but it does not hold for $n = 1$ (see (1*b*) and (1*c*)). For this reason in our proof, see above, we had to start by assuming that (6) hold for $n = 1$, and we then prove, see below, that it holds for $n \geqslant 2$: only using in this process this formula, (76*b*), for $n \geqslant 2$, when it indeed holds. Note that it is the requirement that (6) hold for $n = 1$ that entails hypothesis (B.1*d*), or equivalently (B.1*f*), as a necessary condition for the validity of proposition 2.3; while (6) holds automatically for $n = 0$, see (1*b*). This remark is particularly relevant in view of the existence—see appendix C—of a quite general class of coefficients $a_m^{(\nu)}$ and $b_m^{(\nu)}$ satisfying conditions (B.1*a*) and (B.1*b*) with (B.1*c*) but not the 'initial' conditions (B.1*d*), or equivalently (B.1*f*)—hence failing to satisfy proposition 2.3.

Inserting these two formulae in (75) we get the formula

$$p_{n+1}^{(\nu)}(x) = p_{n+1}^{(\nu-1)}(x) + G_{n+1}^{(\nu)} p_n^{(\nu-1)}(x) + \left(z_{n+1}^{(\nu)} + w_{n+1}^{(\nu)} x\right) p_{n+1}^{(\nu-1)}(x) + \frac{z_{n+1}^{(\nu)} - w_{n+1}^{(\nu)} x}{b_n^{(\nu-1)}} x p_n^{(\nu-1)}(x) \tag{77a}$$

with

$$z_{n+1}^{(\nu)}(x) = \frac{a_n^{(\nu)} b_{n-1}^{(\nu-1)} g_n^{(\nu)} - a_{n-1}^{(\nu-1)} b_n^{(\nu)} g_{n-1}^{(\nu)} + b_{n-1}^{(\nu-1)}\left(b_n^{(\nu)} - b_n^{(\nu-1)}\right)}{b_{n-1}^{(\nu-1)}}, \tag{77b}$$

$$w_{n+1}^{(\nu)} = \frac{b_{n-1}^{(\nu-1)} g_n^{(\nu)} - b_n^{(\nu)} g_{n-1}^{(\nu)}}{b_{n-1}^{(\nu-1)}}, \tag{77c}$$

$$\begin{aligned} G_{n+1}^{(\nu)} = {} & \left(b_n^{(\nu-1)} b_{n-1}^{(\nu-1)}\right)^{-1}\left[b_{n-1}^{(\nu-1)}\left(a_n^{(\nu)} b_n^{(\nu-1)} - a_n^{(\nu-1)} b_n^{(\nu)}\right)\right. \\ & \left. - a_n^{(\nu-1)}\left(a_n^{(\nu)} b_{n-1}^{(\nu-1)} g_n^{(\nu)} - a_{n-1}^{(\nu-1)} b_n^{(\nu)} g_{n-1}^{(\nu)}\right) + b_n^{(\nu)} b_n^{(\nu-1)} g_{n-1}^{(\nu)}\right]. \end{aligned} \tag{77d}$$

Clearly to prove our result one must show that relations (B.1) entail that the two quantities $z_{n+1}^{(\nu)}$, $w_{n+1}^{(\nu)}$ both vanish,

$$z_{n+1}^{(\nu)} = w_{n+1}^{(\nu)} = 0, \tag{78}$$

and moreover that

$$G_{n+1}^{(\nu)} = g_{n+1}^{(\nu)}, \tag{79a}$$

since then (77) yields (6) with n replaced by $n + 1$.

The vanishing of $z_{n+1}^{(\nu)}$ is immediately implied by (B.7*a*), and likewise the vanishing of $w_{n+1}^{(\nu)}$ is implied by (B.1*b*). There remains to prove (79*a*), i.e., (see (77*d*)) the equation

$$\begin{aligned} & b_{n-1}^{(\nu-1)}\left(a_n^{(\nu)} b_n^{(\nu-1)} - a_n^{(\nu-1)} b_n^{(\nu)}\right) - a_n^{(\nu-1)}\left(a_n^{(\nu)} b_{n-1}^{(\nu-1)} g_n^{(\nu)} - a_{n-1}^{(\nu-1)} b_n^{(\nu)} g_{n-1}^{(\nu)}\right) \\ & \qquad = b_n^{(\nu-1)}\left(b_{n-1}^{(\nu-1)} g_{n+1}^{(\nu)} - b_n^{(\nu)} g_{n-1}^{(\nu)}\right). \end{aligned} \tag{79b}$$

Via (B.1*a*) this becomes

$$a_n^{(\nu-1)}\left[a_n^{(\nu)} b_{n-1}^{(\nu-1)} g_n^{(\nu)} - a_{n-1}^{(\nu-1)} b_n^{(\nu)} g_{n-1}^{(\nu)} - b_{n-1}^{(\nu-1)}\left(b_n^{(\nu-1)} - b_n^{(\nu)}\right)\right] = 0, \tag{79c}$$

and its validity is clear from (B.7*a*).

The proof of proposition 2.4 (i.e., of the factorization formula (10)) is again by induction. Clearly (10) holds for $n = 0$ (hence $m = 0$), see (1*b*) and (11*b*). Let us now assume that it holds up to n, and show that it then holds for $n + 1$. Indeed, by using it on the right-hand side of relation (1*a*) with $\nu = m$ we get

$$\begin{aligned} & p_{n+1}^{(m)}(x) = \left[\left(x + a_n^{(m)}\right)\tilde{p}_{n-m}^{(-m)}(x) + b_n^{(m)}\tilde{p}_{n-1-m}^{(-m)}(x)\right] p_m^{(m)}(x), \\ & \qquad m = 0, 1, \ldots, n-1, \end{aligned} \tag{80a}$$

and clearly using the recursion relation (11*a*) the square bracket on the right-hand side of this equation can be replaced by $\tilde{p}_{n+1-m}^{(-m)}(x)$, yielding

$$p_{n+1}^{(m)}(x) = \tilde{p}_{n+1-m}^{(-m)}(x)\, p_m^{(m)}(x), \qquad m = 0, 1, \ldots, n+1. \tag{80b}$$

Note that for $m = n+1$ this formula is an identity, since $\tilde{p}_0^{(-m)}(x) = 1$, see (11*b*); likewise, this formula clearly also holds for $m = n$, provided (8) holds, see (1*a*) with $m = n$ and (11*c*).

But this is just the formula (10) with n replaced by $n+1$.

Remark 4.2. Hypothesis (8) has been used above, in the proof of proposition 2.4, only to prove the validity of the final formula, (80*b*), for $m = n$. Hence one might wonder whether this hypothesis, (8), was redundant, since the validity of the final formula (80*b*) for $m = n$ seems to be implied by (80*a*) with (11*c*) and (11*b*), without the need to invoke (8). But in fact, by setting $m = n$ in the basic recurrence relation (1*a*) it is clear that (11*c*) and (11*b*) hold only provided (8) also holds.

Finally, let us prove proposition 2.8, namely the validity of the factorization formula (18). For $\nu = n$ relation (6) yields

$$p_n^{(n)}(x) = p_n^{(n-1)}(x) + g_n^{(n)}\, p_{n-1}^{(n-1)}(x), \tag{81}$$

and via (9) (with n replaced by $n-1$) this can be rewritten as follows:

$$p_n^{(n)}(x) = \left(x + a_{n-1}^{(n-1)} + g_n^{(n)}\right) p_{n-1}^{(n-1)}(x), \tag{82}$$

clearly entailing (together with the initial condition $p_0^{(0)}(x) = 1$, see (1*b*)), the factorization formula (18).

5. Outlook

As indicated by the Roman numeral appended to its title, the present paper is meant to be the first of a series.

In the next paper of this series we shall focus on the connection of the approach presented herein with results reported in our previous paper [3], and we shall report proofs of certain findings due to Cristophe Smet reported there.

A terse indication of topics that we hope to treat in subsequent papers follows.

The examples involving 'named' polynomials reported in section 3 do not exhaust all such cases encompassed by our treatment: we limited our presentation herein only to some representative examples, to avoid overloading this paper.

The original motivation to arrive at the results reported in this paper arose in the context of the task to prove [3] the Diophantine conjectures proffered in [1, 2] (obtained via the research strategy tersely outlined in appendix D). The Diophantine findings entailed by the results reported in the present paper—in both their avatars, as referring to the zeros of certain monic polynomials belonging to orthogonal families or to the eigenvalues of certain tridiagonal matrices—are more general than those reported in [3], and it is likely that even more general results could be found by extending the approach employed in the present paper, see for instance the results of appendix A or imagine variants of the second recursion relation (6).

Another point of view deserving future scrutiny is the interpretation of the nonlinear relations (4) satisfied by the quantities $A_n^{(\nu)}$ and $\alpha^{(\nu)}$ (or equivalently of relations (85) satisfied by the coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$) as discrete evolution equations satisfied by these quantities—considered as dependent variables, with n and ν playing correspondingly the role of independent variables: nonlinear evolution equations which are integrable inasmuch as they play the role of compatibility conditions of the linear relations (1) and (6) (interpretable in this context as a Lax pair).

And, last but not least, the classes of (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ identified in this paper deserve further study—for instance to exhibit their orthogonality properties and their relations (if any) to (possibly generalized) hypergeometric functions.

Acknowledgment

It is a pleasure to thank R Askey for useful correspondence and for providing us with a copy of his paper [5].

Appendix A

In this appendix A we report the conditions on the coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ that are necessary and sufficient to guarantee that the polynomials $p_n^{(\nu)}(x)$ satisfy—in addition to (1)—the following recursion relation:

$$p_n^{(\nu)}(x) = u_n^{(\nu)} p_n^{(\nu-1)}(x) + \left(g_n^{(\nu)} + h_n^{(\nu)} x\right) p_{n-1}^{(\nu-1)}(x), \tag{A.1}$$

which is a more general version of (6) (to which it reduces for $u_n^{(\nu)} = 1$ and $h_n^{(\nu)} = 0$). They read

$$a_n^{(\nu)} u_n^{(\nu)} - a_n^{(\nu-1)} u_{n+1}^{(\nu)} = g_{n+1}^{(\nu)} - g_n^{(\nu)} - \left(a_n^{(\nu)} - a_{n-1}^{(\nu-1)}\right) h_n^{(\nu)}, \tag{A.2a}$$

$$b_{n-1}^{(\nu-1)} g_n^{(\nu)} - b_n^{(\nu)} g_{n-1}^{(\nu)} = \left(a_n^{(\nu)} - a_{n-1}^{(\nu-1)}\right) b_{n-1}^{(\nu-1)} h_n^{(\nu)}, \tag{A.2b}$$

$$\left(a_n^{(\nu)} - a_{n-1}^{(\nu-1)}\right) g_n^{(\nu)} + b_n^{(\nu)} u_{n-1}^{(\nu)} - b_n^{(\nu-1)} u_n^{(\nu)} = -a_{n-1}^{(\nu)} \left(a_n^{(\nu)} h_n^{(\nu)} - a_{n-1}^{(\nu-1)} h_{n-1}^{(\nu)}\right), \tag{A.2c}$$

$$b_{n-1}^{(\nu-1)} h_n^{(\nu)} - b_n^{(\nu)} h_{n-1}^{(\nu)} = 0, \tag{A.2d}$$

$$u_{n+1}^{(\nu)} - u_n^{(\nu)} + b_{n-1}^{(\nu-1)} h_{n+1}^{(\nu)} = 0 \tag{A.2e}$$

with the 'initial' conditions

$$g_1^{(\nu)} = a_0^{(\nu)} - u_1^{(\nu)} a_0^{(\nu-1)}, \tag{A.2f}$$

$$u_0^{(\nu)} = 1, \tag{A.2g}$$

$$h_1^{(\nu)} = 1 - u_1^{(\nu)}. \tag{A.2h}$$

It is clear that these conditions reduce to (B.1) if $u_n^{(\nu)} = 1$ and $h_n^{(\nu)} = 0$; and their derivation is sufficiently analogous to that of (B.1) (see section 4 and appendix B) that we feel justified in leaving this task to the diligent reader.

Appendix B

In this appendix B we obtain some relations satisfied by the quantities $a_n^{(\nu)}, b_n^{(\nu)}, g_n^{(\nu)}$, as entailed by relations (5) and (7) with (4). These formulae are used in section 4 to prove our main results (see section 2). We also report some other relations that might eventually prove useful to identify other assignments of the parameters $a_n^{(\nu)}$ and $b_n^{(\nu)}$ yielding via (1) novel classes of (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ satisfying our main results, see section 2.

The main relations read as follows:

$$a_n^{(\nu)} - a_n^{(\nu-1)} = g_{n+1}^{(\nu)} - g_n^{(\nu)}, \tag{B.1a}$$

$$b_{n-1}^{(\nu-1)} g_n^{(\nu)} - b_n^{(\nu)} g_{n-1}^{(\nu)} = 0, \tag{B.1b}$$

with

$$g_n^{(\nu)} = -\frac{b_n^{(\nu)} - b_n^{(\nu-1)}}{a_n^{(\nu)} - a_{n-1}^{(\nu-1)}}, \tag{B.1c}$$

and the 'initial' condition

$$g_1^{(\nu)} = a_0^{(\nu)} - a_0^{(\nu-1)} \tag{B.1d}$$

entailing via (B.1*c*) (with $n = 1$)

$$b_1^{(\nu)} - b_1^{(\nu-1)} + \left(a_0^{(\nu)} - a_0^{(\nu-1)}\right)\left(a_1^{(\nu)} - a_0^{(\nu-1)}\right) = 0 \tag{B.1e}$$

and via (B.1*a*) (with $n = 0$)

$$g_0^{(\nu)} = 0. \tag{B.1f}$$

The fact that these relations correspond to (5) and (7) with (4) is plain: indeed (B.1) follows immediately from (5*a*) and (7), while (B.1*b*) follows from (5*b*) and (7) via (4*a*). To prove (B.1*c*) we conveniently set

$$b_n^{(\nu)} = \beta_n^{(\nu)} g_n^{(\nu)}, \tag{B.2}$$

with (as implied by (5*b*) and (7))

$$\beta_n^{(\nu)} = A_n^{(\nu)} - A_{n-1}^{(\nu-1)} + \alpha^{(\nu)}. \tag{B.3}$$

We then note that (as implied by (B.1*b*) and (B.2))

$$a_n^{(\nu)} - a_{n-1}^{(\nu-1)} = -\beta_n^{(\nu)} + \beta_{n+1}^{(\nu)}, \tag{B.4}$$

and this relation becomes an identity via (5*a*) and (B.3).

The correspondence via (7) of the initial condition (B.1*f*) with (4*b*) is as well plain.

An additional formula implied by these relations, also used in section 4, reads

$$a_n^{(\nu)} b_{n-1}^{(\nu-1)} g_n^{(\nu)} - a_{n-1}^{(\nu-1)} b_n^{(\nu)} g_{n-1}^{(\nu)} = b_{n-1}^{(\nu-1)}\left(b_n^{(\nu-1)} - b_n^{(\nu)}\right). \tag{B.5a}$$

Indeed via (B.1) this relation can be rewritten as follows:

$$b_{n-1}^{(\nu-1)}\left[a_n^{(\nu)} - a_{n-1}^{(\nu-1)}\right] g_n^{(\nu)} = b_{n-1}^{(\nu-1)}\left[b_n^{(\nu-1)} - b_n^{(\nu)}\right], \tag{B.5b}$$

and clearly this becomes an identity via (B.1*c*).

We moreover note (and prove at the end of this appendix B) that, if the quantities $\beta_n^{(\nu)}$ defined by (B.2) satisfy the (nonlinear) relations

$$\beta_{n+1}^{(\nu+1)} g_n^{(\nu+1)} = \beta_n^{(\nu)} g_n^{(\nu)} \tag{B.6a}$$

with

$$g_n^{(\nu)} = \gamma^{(\nu-1)} - \gamma^{(\nu-n-1)} + \sum_{m=1}^{n} \left[\beta_m^{(\nu+m-n)} - \beta_m^{(\nu-1+m-n)}\right] \tag{B.6b}$$

where the quantities $\gamma^{(\nu)}$ are *a priori* arbitrary (they of course may only depend on the upper index ν), then the following assignment in terms of these quantities $\beta_n^{(\nu)}$ of the coefficients $a_n^{(\nu)}$:

$$a_n^{(\nu)} = \gamma^{(\nu)} + \beta_{n+1}^{(\nu+1)} - g_{n+1}^{(\nu+1)}, \tag{B.6c}$$

together with assignment (B.4) of the quantities $b_n^{(\nu)}$, satisfy all conditions (B.1) required for the validity of the preceding proposition 2.3 (with $g_n^{(\nu)}$ defined of course by (B.6*b*)). Note that in (B.6), as well as hereafter, we adopt the standard convention according to which a sum vanishes if its lower limit exceeds its upper limit.

Likewise, if we set (as entailed by (B.1*a*) with (B.1*d*))

$$g_n^{(\nu)} = \sum_{m=1}^{n} \left[a_{m-1}^{(\nu)} - a_{m-1}^{(\nu-1)}\right], \tag{B.7a}$$

and the coefficients $a_n^{(\nu)}$ satisfy the (nonlinear) relation

$$g_n^{(\nu)}\left[\eta^{(\nu)} + a_{n-1}^{(\nu-1)} + g_n^{(\nu)}\right] = g_n^{(\nu+1)}\left[\eta^{(\nu+1)} + a_n^{(\nu+1)} + g_n^{(\nu+1)}\right], \tag{B.7b}$$

with the quantities $\eta^{(\nu)}$ again independent of the index n but otherwise *a priori* arbitrary, then the following assignment of the coefficients $b_n^{(\nu)}$ (together with assignment (B.9*a*) of the coefficients $g_n^{(\nu)}$) satisfy all conditions (B.1) required for the validity of the preceding proposition 2.3:

$$b_n^{(\nu)} = g_n^{(\nu)}\left[\eta^{(\nu)} + a_{n-1}^{(\nu)} + g_{n-1}^{(\nu)}\right], \tag{B.7c}$$

or equivalently (see (B.7*b*))

$$b_n^{(\nu)} = g_n^{(\nu+1)}\left[\eta^{(\nu+1)} + a_n^{(\nu+1)} + g_n^{(\nu+1)}\right]. \tag{B.7d}$$

Remark B.1. As can be easily verified, the following two four-parameter assignments of the quantities $\beta_n^{(\nu)}$ satisfy the nonlinear conditions (B.6):

$$\beta_n^{(\nu)} = \beta_{00} + \beta_{10} n + \beta_{01}\nu + \beta_{20} n^2, \tag{B.8a}$$

$$\beta_n^{(\nu)} = \beta_{00} + \beta_{10} n + \beta_{01}\nu + \beta_{02} \nu^2, \tag{B.8b}$$

with the coefficients β_{jk} denoting arbitrary numbers. But, via (B.6) with (B.2) and (B.6*c*), both these assignments yield the uninteresting result $g_n^{(\nu)} = b_n^{(\nu)} = 0$ and $a_n^{(\nu)} = a_n$. The same uninteresting outcome is produced by the assignment $\beta_n^{(\nu)} = \beta_n$, with the coefficients β_n depending arbitrarily on the index n but being independent of ν, an assignment that clearly satisfies (B.6*a*) trivially. And uninteresting outcomes are obviously also produced (see (B.7*c*) and (B.7*d*)) by any solution that equates to zero both sides of the equations (B.6*a*) or (B.7*b*).

Next we provide another expression (additional to (B.1*c*)) for $g_n^{(\nu)}$, which is clearly implied by (B.1*b*) with (B.1*d*):

$$g_n^{(\nu)} = \left[a_0^{(\nu)} - a_0^{(\nu-1)}\right] \prod_{m=1}^{n-1} \left[\frac{b_{m+1}^{(\nu)}}{b_m^{(\nu-1)}}\right], \qquad n = 1, 2, \ldots. \tag{B.9}$$

Here and hereafter we adopt the standard convention according to which a product has unit value if its lower limit exceeds its upper limit.

Note that, if (8) holds, some care must be used when employing this formula for $\nu = 2, \ldots, n$, because one factor of the product will then take the (*a priori* indeterminate) value $0/0$.

Likewise solving for $b_n^{(\nu)}$ the recursion (B.1*b*) by keeping fixed $\mu = n - \nu$ one easily gets

$$b_n^{(\nu)} = b_1^{(\nu+1-n)} \prod_{m=1}^{n-1} \left[\frac{g_{m+1}^{(m-n+\nu+1)}}{g_m^{(m-n+\nu+1)}}\right]. \tag{B.10}$$

And in a similar vein by setting $\nu = \mu + \ell$ in (B.1*a*) and summing over the integer index ℓ we get

$$a_n^{(\mu+m)} = a_n^{(\mu)} + \sum_{\ell=1}^{m} \left(g_{n+1}^{(\mu+\ell)} - g_n^{(\mu+\ell)}\right), \tag{B.11}$$

where of course m is a nonnegative integer.

One can also rewrite (B.1*c*) as follows,

$$a_n^{(\nu)} - a_{n-1}^{(\nu-1)} = -\frac{b_n^{(\nu)} - b_n^{(\nu-1)}}{g_n^{(\nu)}}, \tag{B.12}$$

and by summing this expression over n with $n - \nu = \mu$ kept fixed one obtains the additional relation

$$a_n^{(\nu)} = a_0^{(\nu-n)} + \sum_{m=1}^{n} \left[\frac{b_m^{(\nu-1+m-n)} - b_m^{(\nu+m-n)}}{g_m^{(\nu+m-n)}}\right]. \tag{B.13}$$

Finally, let us obtain relations (B.6*a*). Subtracting (B.1*a*) from (B.4) we get the relation

$$a_n^{(\nu-1)} - \beta_{n+1}^{(\nu)} + g_{n+1}^{(\nu)} = a_{n-1}^{(\nu-1)} - \beta_n^{(\nu)} + g_n^{(\nu)}, \tag{B.14}$$

clearly entailing (B.6*c*), that is thereby proven. Then the insertion of this expression, (B.6*c*), of $a_n^{(\nu)}$ in (B.1*a*) yields

$$g_{n+1}^{(\nu+1)} - g_n^{(\nu)} = \gamma^{(\nu)} - \gamma^{(\nu-1)} + \beta_{n+1}^{(\nu+1)} - \beta_{n+1}^{(\nu)}, \tag{B.15a}$$

and by summing this equation over n with $n - \nu = \mu$ kept fixed we obtain the formula

$$g_n^{(\nu)} = g_0^{(\nu-n)} + \gamma^{(\nu-1)} - \gamma^{(\nu-n-1)} + \sum_{m=1}^{n} \left[\beta_m^{(\nu+m-n)} - \beta_m^{(\nu-1+m-n)}\right], \tag{B.15b}$$

which, via (B.1*f*), yields (B.6*b*).

Appendix C

In this appendix C we report a solution, involving an arbitrary function, of conditions (B.1*a*) and (B.1*b*) with (B.1*c*), which also satisfies the symmetry property (13):

$$a_n^{(\nu)} = f(2n - \nu) + f(2n - \nu + 1), \tag{C.1a}$$

$$b_n^{(\nu)} = -f(2n - \nu) f(2n - \nu - 1), \tag{C.2b}$$

$$g_n^{(\nu)} = -f(2n - \nu), \tag{C.3c}$$

with $f(z)$ an arbitrary function. But it is plain that this solution disappears altogether ($f(z) = 0$) if one requires it to satisfy either the additional 'initial' condition (B.1*d*) (or equivalently (B.1*f*)) also required for the validity of proposition 2.3 or the hypothesis (8) required for the validity of proposition 2.6.

Appendix D

The original strategy—as fully described for instance in [4]—to arrive at the Diophantine findings that provided the motivation for the developments reported in this paper can be outlined as follows. (i) Identify an integrable dynamical system. (ii) Modify it so that it becomes isochronous. (iii) Identify an equilibrium configuration of the isochronous system.

(iv) Investigate, via the standard linearization technique (the theory of small oscillations around equilibria) the behaviour of the isochronous system near its equilibrium configuration, which is then characterized by a set of basic oscillation frequencies whose values are provided by the eigenvalues of a matrix obtained from the equations of motion and evaluated at the equilibrium values of the dependent variables. (v) Observe that—because the isochronous nature of the dynamical system under consideration must also characterize its behaviour around equilibrium—all these basic frequencies of oscillation must be integer multiples of a basic frequency. (vi) Infer that all the eigenvalues of the matrix characterizing the behaviour around equilibrium must be integers (up to a common rescaling). This fact—that all the eigenvalues of a matrix, of arbitrary order and of reasonably neat appearance, are integers—constitute the Diophantine finding (which is nontrivial provided the similarity transformation diagonalizing the matrix in question is not obvious).

In this appendix D we provide—quite tersely—two (new) examples of this procedure.

D.1. First example: Toda many-body model

An integrable dynamical system:

$$\eta_n' = \xi_n - \xi_{n-1}, \qquad \xi_n' = \xi_n(\eta_{n+1} - \eta_n). \tag{D.1}$$

These are the equations of motion (in the version more convenient for our purposes) of the classical Toda model [14, 15], whose integrability was noted by Henon [16] and demonstrated by Flaschka [17, 18] and Manakov [19].

Free-end boundary conditions:

$$\xi_0 = \eta_{N+1} = 0. \tag{D.2}$$

The trick:

$$y_n(t) = \exp(\mathrm{i}t)\eta_n(\tau), \qquad x_n(t) = \exp(2\mathrm{i}t)\xi_n(\tau), \qquad \tau = \mathrm{i}[1 - \exp(\mathrm{i}t)]. \tag{D.3}$$

The isochronous version:

$$\dot{y}_n - \mathrm{i}\omega \dot{y}_n = x_n - x_{n-1}, \qquad \dot{x}_n - 2\mathrm{i}\omega x_n = x_n(y_{n+1} - y_n), \qquad x_0 = y_{N+1} = 0. \tag{D.4}$$

Equilibrium configuration (satisfying the free-end boundary conditions):

$$x_n(t) = \bar{x}_n = n(2N + 1 - n), \qquad y_n(t) = \bar{y}_n = 2\mathrm{i}(N + 1 - n). \tag{D.5}$$

Small oscillations around equilibrium:

$$x_n(t) = \bar{x}_n + \epsilon u_n(t), \qquad y_n(t) = \bar{y}_n + \epsilon w_n(t), \qquad \epsilon \approx 0. \tag{D.6}$$

The linearized equations of motion:

$$\dot{w}_n - \mathrm{i}w_n = u_n - u_{n-1}, \qquad u_0 = 0, \tag{D.7a}$$

$$\dot{u}_n = n(2N + 1 - n)(w_{n+1} - w_n), \qquad w_{N+1} = 0, \tag{D.7b}$$

$$\ddot{w}_n - \mathrm{i}\dot{w}_n - n(2N + 1 - n)w_{n+1} + 2[N(2n - 1) - n + 1]w_n \\ - (n - 1)(2N - n + 2)w_{n-1} = 0, \qquad w_{N+1} = 0. \tag{D.7c}$$

The basic oscillations:

$$w_n(t) = \tilde{w}_n \exp(\mathrm{i}\lambda t). \tag{D.8}$$

The eigenvalue problem determining the eigenfrequencies λ:

$$\lambda(\lambda - 1)\tilde{\xi}_n + n(2N + 1 - n)\tilde{w}_{n+1} - 2[N(2n - 1) - n + 1]\tilde{w}_n \\ + (n - 1)(2N - n + 2)\tilde{w}_{n-1} = 0, \qquad \tilde{w}_{N+1} = 0. \tag{D.9}$$

Diophantine finding: setting $N = \mu$, defining the $n \times n$ matrix $L^{(\mu)}$ as follows,

$$L^{(\mu)}_{m,m} = m(2\mu - m + 1) + (m - 1)(2\mu - m + 2)], \tag{D.10a}$$

$$L^{(\mu)}_{m,m-1} = -(m - 1)(2\mu - m + 2), \tag{D.10b}$$

$$L^{(\mu)}_{m,m+1} = -n(2\mu - m + 1), \tag{D.10c}$$

one concludes that the solution of the following polynomial equation of degree $2n$ in λ must have rational solutions,

$$\det[\lambda(\lambda - 1) - L^{(n)}] = 0. \tag{D.11}$$

Hence, setting $z = \lambda(\lambda - 1)$ so that $\lambda = (1 \pm \sqrt{1 + 4z})/2$, one infers that $1 + 4z$ must be the square of a rational number, say $1 + 4z = (4m - 1)^2$ hence $z = 2m(2m - 1)$ with m rational. Indeed one finds by direct calculation

$$\begin{vmatrix} z-4 & 4 \\ 4 & q-10 \end{vmatrix} = (z-2)(z-12), \tag{D.12a}$$

$$\begin{vmatrix} z-6 & 6 & 0 \\ 6 & z-16 & 10 \\ 0 & 10 & z-22 \end{vmatrix} = (z-2)(z-12)(z-30), \tag{D.12b}$$

$$\begin{vmatrix} z-8 & 8 & 0 & 0 \\ 8 & z-22 & 14 & 0 \\ 0 & 14 & z-32 & 18 \\ 0 & 0 & 18 & z-38 \end{vmatrix} = (z-2)(z-12)(z-30)(z-56), \tag{D.12c}$$

$$\begin{vmatrix} z-10 & 10 & 0 & 0 & 0 \\ 10 & z-28 & 18 & 0 & 0 \\ 0 & 18 & z-42 & 24 & 0 \\ 0 & 0 & 24 & z-52 & 28 \\ 0 & 0 & 0 & 28 & z-58 \end{vmatrix} = (z-2)(z-12)(z-30)(z-56)(z-90). \tag{D.12d}$$

One therefore sees that the numbers m are in fact integers, and infers that, if one defines the family of polynomials $P^{(\mu)}_n(z)$, of degree n, via the formula

$$\det[z - L^{(\mu)}] = P^{(\mu)}_n(z), \tag{D.13}$$

there holds the Diophantine property

$$P^{(n)}_n(z) = \prod_{m=1}^{n} [z - 2m(2m - 1)]. \tag{D.14}$$

Indeed the tridiagonal character of the $n \times n$ matrix $L^{(\mu)}$, see (D.10), entails that the family of (monic, orthogonal) polynomials (D.13) is characterized by the recursion relation

$$P^{(\mu)}_{n+1}(x) = (x - 2n^2 + 4n\mu + 2\mu)P^{(\mu)}_n(x) - n^2(2\mu - n + 1)^2 P^{(\mu)}_{n-1}(z), \tag{D.15a}$$

$$P^{(\nu)}_{-1}(x) = 0, \qquad P^{(\nu)}_0(x) = 1, \tag{D.15b}$$

and it is therefore immediately seen that they coincide with the polynomials $p_n^{(\nu)}(x)$ of our previous paper [3] up to the identification $\nu = 2\mu + 1$, so that the Diophantine factorization (D.14) coincides with the Smet formula (see equation (59) of our previous paper [3])

$$p_n^{(2n+1)}(x) = \prod_{m=1}^{n}[x - 2m(2m-1)]. \tag{D.16}$$

A proof of this formula is actually provided in the following paper of this series [21].

D.2. Second example: Bruschi–Ragnisco–Levi many-body model

An integrable dynamical system:

$$\zeta_m' = \zeta_m(\zeta_{m-1} - \zeta_{m+1}), \qquad m = 1, \ldots, M, \qquad \zeta_m \equiv \zeta_m(\tau). \tag{D.17}$$

This is the scalar version of equation (5.4.3–13a) of [13] (it will eventually be worthwhile to consider other equations of this hierarchy [20]).

Free-end boundary conditions:

$$\zeta_0 = \zeta_{M+1} = 0. \tag{D.18}$$

The trick:

$$z_m(t) = \exp(\mathrm{i}t)\zeta_m(\tau), \qquad \tau = \mathrm{i}[1 - \exp(\mathrm{i}t)]. \tag{D.19}$$

The isochronous version:

$$\dot{z}_m - \mathrm{i}z_m = z_m(z_{m-1} - z_{m+1}), \qquad m = 1, \ldots, M, \tag{D.20a}$$

$$z_0 = z_{M+1} = 0. \tag{D.20b}$$

A convenient ansatz:

$$M = 2N - 1, \qquad z_{2n-1} = x_n, z_{2n} = y_n, \qquad n = 0, 1, \ldots, N, N+1; \tag{D.21a}$$

$$m = 2n - 1 : \dot{x}_n - \mathrm{i}x_n = x_n(y_{n-1} - y_n), \qquad n = 1, \ldots, N, \qquad y_0 = 0, \tag{D.21b}$$

$$m = 2n : \dot{y}_n - \mathrm{i}y_n = y_n(x_n - x_{n+1}), \qquad n = 1, \ldots, N, \qquad x_{N+1} = 0. \tag{D.21c}$$

Equilibrium configuration (satisfying the free-end boundary conditions):

$$x_n(t) = \bar{x}_n = \mathrm{i}(n - N - 1), \qquad y_n(t) = \bar{y}_n = \mathrm{i}n. \tag{D.22}$$

Small oscillations around equilibrium:

$$x_n(t) = \bar{x}_n + \epsilon\xi_n(t), \qquad y_n(t) = \bar{y}_n + \epsilon\eta_n(t), \qquad \epsilon \approx 0. \tag{D.23}$$

The linearized equations of motion:

$$\dot{\xi}_n = \mathrm{i}(n - N - 1)(\eta_{n-1} - \eta_n), \qquad \xi_{N+1} = 0, \tag{D.24a}$$

$$\dot{\eta}_n = \mathrm{i}n(\xi_n - \xi_{n+1}), \qquad \eta_0 = 0; \tag{D.24b}$$

$$\ddot{\xi}_n = (N + 1 - n)[(n-1)\xi_{n-1} - (2n-1)\xi_n + n\xi_{n+1}], \qquad \xi_0 = \xi_{N+1} = 0. \tag{D.24c}$$

The basic oscillations:

$$\xi_n(t) = \tilde{\xi}_n \exp(\mathrm{i}\lambda t). \tag{D.25}$$

The eigenvalue problem determining the eigenfrequencies λ:

$$\lambda^2\tilde{\xi}_n + (N + 1 - n)[(n-1)\tilde{\xi}_{n-1} - (2n-1)\tilde{\xi}_n + n\tilde{\xi}_{n+1}] = 0, \qquad \tilde{\xi}_0 = \tilde{\xi}_{N+1} = 0. \tag{D.26}$$

Diophantine finding: setting $N = \nu$, defining now the $n \times n$ matrix $M^{(\nu)}$ as follows,

$$M^{(\nu)}_{m,m+1} = -m(\nu + 1 - m), \qquad m = 1, \ldots, n-1, \tag{D.27a}$$

$$M^{(\nu)}_{m,m} = (2m - 1)(\nu + 1 - m), \qquad m = 1, \ldots, n, \tag{D.27b}$$

$$M^{(\nu)}_{m,m-1} = -(m - 1)(\nu + 1 - m), \qquad m = 2, \ldots, n, \tag{D.27c}$$

one finds

$$\det[x - M^{(n)}] = \prod_{m=1}^{n} (x - m^2). \tag{D.28}$$

Examples:

$$\begin{vmatrix} x-2 & 2 \\ 1 & x-3 \end{vmatrix} = (x-1)(x-4), \tag{D.29a}$$

$$\begin{vmatrix} x-3 & 3 & 0 \\ 2 & x-6 & 4 \\ 0 & 2 & x-5 \end{vmatrix} = (x-1)(x-4)(x-9), \tag{D.29b}$$

$$\begin{vmatrix} x-4 & 4 & 0 & 0 \\ 3 & x-9 & 6 & 0 \\ 0 & 4 & x-10 & 6 \\ 0 & 0 & 3 & x-7 \end{vmatrix} = (x-1)(x-4)(x-9)(x-16), \tag{D.29c}$$

$$\begin{vmatrix} x-5 & 5 & 0 & 0 & 0 \\ 4 & x-12 & 8 & 0 & 0 \\ 0 & 6 & x-15 & 3 & 0 \\ 0 & 0 & 6 & x-14 & 8 \\ 0 & 0 & 0 & 4 & x-9 \end{vmatrix} = (x-1)(x-4)(x-9)(x-16)(x-25), \tag{D.29d}$$

$$\begin{vmatrix} x-6 & 6 & 0 & 0 & 0 & 0 \\ 5 & x-15 & 10 & 0 & 0 & 0 \\ 0 & 8 & x-20 & 12 & 0 & 0 \\ 0 & 0 & 9 & x-21 & 12 & 0 \\ 0 & 0 & 0 & 8 & x-18 & 10 \\ 0 & 0 & 0 & 0 & 5 & x-11 \end{vmatrix} = (x-1)(x-4)(x-9)(x-16)(x-25)(x-36), \tag{D.29e}$$

$$\begin{vmatrix} x-7 & 7 & 0 & 0 & 0 & 0 & 0 \\ 6 & x-18 & 12 & 0 & 0 & 0 & 0 \\ 0 & 10 & x-25 & 15 & 0 & 0 & 0 \\ 0 & 0 & 12 & x-28 & 16 & 0 & 0 \\ 0 & 0 & 0 & 12 & x-27 & 15 & 0 \\ 0 & 0 & 0 & 0 & 10 & x-22 & 12 \\ 0 & 0 & 0 & 0 & 0 & 6 & x-13 \end{vmatrix} = (x-1)(x-4)(x-9)(x-16)(x-25)(x-36)(x-49). \tag{D.29f}$$

The corresponding orthogonal polynomials:

$$\check{p}^{(\nu)}_n(x) = \det[x - M^{(\nu)}], \tag{D.30}$$

$$\check{p}_{n+1}^{(\nu)}(x) = \left(x + a_n^{(\nu)}\right)\check{p}_n^{(\nu)}(x) + b_n^{(\nu)}\check{p}_{n-1}^{(\nu)}(x), \tag{D.31a}$$

$$\check{p}_{-1}^{(\nu)}(x) = 0, \qquad \check{p}_0^{(\nu)}(x) = 1, \tag{D.31b}$$

$$a_n^{(\nu)} = -(2n+1)(\nu - n), \tag{D.31c}$$

$$b_n^{(\nu)} = n^2(\nu - n)(\nu + 1 - n). \tag{D.31d}$$

Identification with the results of sections 2 and 3 (see in particular section 3.1):

$$\check{p}_n^{(\nu)}(x) = p_n^{(\nu)}(x; k_1, k_2, k_3, k_4, k_5), \tag{D.32a}$$
$$k_1 = -\tfrac{1}{6}, \qquad k_2 = -\tfrac{1}{2}, \qquad k_3 = \tfrac{2}{3}, \qquad k_4 = k_5 = 0. \tag{D.32b}$$

Then the remark 3.2 demonstrates the validity of the Diophantine finding (D.28).

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